Journal of Statistical Physics, Vol. 52, Nos. 3/4, 1988

Correlation Inequalities for Two-Component Hypercubic φ^4 Models

José L. Soria[1]

Received February 3, 1988; revision April 13, 1988

A collection of new and already known correlation inequalities is found for a family of two-component hypercubic φ^4 models, using techniques of duplicated variables, rotated correlation inequalities, and random walk representation. Among the interesting new inequalities are: rotated very special Dunlop–Newman inequality $\langle \varphi_{1x}^2; \varphi_{1z}^2 + \varphi_{2z}^2 \rangle \geq 0$, rotated Griffiths I inequality $\langle \varphi_{1x}\varphi_{1y}; \varphi_{1z}^2 - \varphi_{2z}^2 \rangle \geq 0$, and anti-Lebowitz inequality $u_4^{1111} \geq 0$.

KEY WORDS: Correlation inequalities; hypercubic symmetry; rotated inequalities; random walk representation.

1. INTRODUCTION

In this paper we study a class of two-component φ^4 models with hypercubic internal symmetry, i.e., interacting Hamiltonian

$$H = \sum_x \{\lambda_1[\varphi_{1x}^4 + \varphi_{2x}^4] + \lambda_2 \varphi_{1x}^2 \varphi_{2x}^2\} \tag{1.1}$$

This work was motivated by the attempt to prove the conjecture, suggested by perturbation theory, that the physical mass for a pure $\varphi_1^2\varphi_2^2$ model is strictly positive even as the bare mass tends to zero (i.e., there is no critical point).[1]

Let $\mathbf{L} \in \mathbf{Z}^d$ be a finite lattice. To each point $x \in \mathbf{L}$ is associated a field $\boldsymbol{\varphi}_x = (\varphi_{1x}, \varphi_{2x})$ where each component is a real random variable. The

[1] Courant Institute of Mathematical Sciences, New York University, New York, New York 10012. Address beginning September 1988: Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, UNAM, Apartado Postal 20–726, 01000 México D.F. México.

0022-4715/88/0800-0711$06.00/0

collection of these random variables, $\varphi = \{\varphi_x : x \in \mathbf{L}\}$, is distributed according to the probability measure

$$Z^{-1} \exp\left[\frac{1}{2}(\varphi, J\varphi)\right] \times \prod_x \exp\left(-\lambda_1[\varphi_{1x}^4 + \varphi_{2x}^4] - \lambda_2 \varphi_{1x}^2 \varphi_{2x}^2 - \frac{a_1}{2}\varphi_{1x}^2 - \frac{a_2}{2}\varphi_{2x}^2 - b\varphi_{1x}\varphi_{2x}\right) d\varphi_{1x}\, d\varphi_{2x} \tag{1.2}$$

where Z is a normalization constant; J is an off-diagonal symmetric matrix with $J_{xy}^1 = J_{xy}^2 \geqslant 0$ for all x, $y \in \mathbf{L}$; $\lambda_1 > 0$; $\lambda_2 \geqslant -2\lambda_1$; a_1, a_2, $b \in \mathbf{R}$; and when $\lambda_1 = 0$ then we require $a_1, a_2 > 0$. For future use we define $R = \lambda_2/\lambda_1 \in [-2, \infty]$. Note that $R = 0$, $b = 0$ represents two decoupled one-component φ^4 models; $R = 2$, $a_1 = a_2$, $b = 0$ represents the isotropic $|\varphi|^4$ models; and $R = \infty$, $b = 0$ represents the pure $\varphi_1^2 \varphi_2^2$ models.

The main goal of this paper is to prove correlation inequalities for models with joint probability measure (1.2) that would allow us to study the critical behavior. We use three techniques: duplicate variables (discussed in Section 2), the process of rotated correlation inequalities (discussed in Section 3) and the random walk representation (discussed in Sections 4–6).

The new results of this work are summarized in Table I. In addition, we rederive by new methods some previously known inequalities.

2. COMPONENTWISE AND (ρ, θ)-TYPE INEQUALITIES

Here we classify correlation inequalities that use the technique of duplicate variables for two-component systems (according to the method of the proof) as follows:

(i) Componentwise type (Fourier analysis on Z_2). Bricmont–Monroe inequalities[2] (we state this result following ref. 2, without proof).

(ii) (ρ, θ) type [Fourier analysis on $U(1)$]. Dunlop–Newman inequalities (we sketch the proof for them).

[2] These inequalities are usually called Monroe inequalities, since he introduced and proved them. However, we call them by the name of Bricmont–Monroe since we use the general version of Bricmont.

Table I. Inequalities Proven for Two-Component Systems with Joint Probability Measure Given by (1.2)

Name	Inequality	Assumptions	Method of proof	Location
rBM-I	$\langle \varphi_1^A; \varphi_1^B \rangle \geqslant 0$			
rBM-II	$\langle \varphi_2^A; \varphi_2^B \rangle \geqslant 0$	$b \leqslant 0;\ R \in [-2, 0]$	Duplicate variables	Proposition 2.2
rBM-III	$\langle \varphi_1^A; \varphi_2^B \rangle \geqslant 0$			
rotvsD-N	$\langle \varphi_{1x}^2; \varphi_{1z}^2 + \varphi_{2z}^2 \rangle \geqslant 0$	$b=0;\ a_1=a_2;\ R \in [0, 2]$	Rotated vsD-N	Proposition 3.1
rotG-I	$\langle \varphi_{1x}\varphi_{1y}; \varphi_{1z}^2 - \varphi_{2z}^2 \rangle \geqslant 0$	$b=0;\ a_1=a_2;\ R \in [-2, 0]$	Rotated G-I	Proposition 3.2
	$u_4^{1111} \leqslant 0$	$b=0;\ a_1 \leqslant a_2;\ R=2$	R.W.	Eq. (6.5)
	$u_4^{1111} \geqslant 0$	$b=0;\ R=\infty$	R.W.	Eq. (6.7)
	$u_4^{1111} - \frac{1}{3}(u_4^{1122} + u_4^{1212} + u_4^{1221}) \leqslant 0$	$b=0;\ a_1=a_2;\ R \leqslant 6$	Rotated u_4^{1122}	Eq. (3.17)

2.1. Componentwise Inequalities

As is well known, Griffiths' second inequality, in contrast with Griffiths' first, has not been extended for general N. However, there are some positive results for $N=2$ by Monroe[4] and for $N=3, 4$ by Dunlop[5] and Kunz *et al.*[6]

Theorem 2.1. (Bricmont–Monroe inequalities[2,4,7]). Let $(\boldsymbol{\varphi}_x)=\{(\varphi_{1x}, \varphi_{2x}); x \in \mathbf{L}\}$ be a set of classical spins with partition function Z in a finite lattice $L \in \mathbf{Z}^d$,

$$Z=\int \exp\left(\sum_K J_K^1 \varphi_1^K + J_K^2 \varphi_2^K\right) \prod_{x \in L} d\nu_x(\varphi_{1x}, \varphi_{2x}) \tag{2.1}$$

where the *a priori* measure is given by

$$d\nu_x(\varphi_{1x}, \varphi_{2x}) = \exp\left(-\sum_{r,p=0}^{\infty} a_{rp} \varphi_{1x}^{2r} \varphi_{2x}^{2p}\right) d\nu_{1x}(\varphi_{1x})\, d\nu_{2x}(\varphi_{2x}) \tag{2.2}$$

with $a_{rp} \geqslant 0$ if both r and p are different from zero. J_K^1, $J_K^2 \geqslant 0$ for all multi-indices K and $d\nu_{1x}$, $d\nu_{2x}$ even Borel measures for all $x \in \mathbf{L}$. Then for any multi-indices A and B

$$\text{(BM-I)} \qquad \langle \varphi_1^A; \varphi_1^B \rangle \geqslant 0 \tag{2.3}$$

$$\text{(BM-II)} \qquad \langle \varphi_2^A; \varphi_2^B \rangle \geqslant 0 \tag{2.4}$$

$$\text{(BM-III)} \qquad \langle \varphi_1^A; \varphi_2^B \rangle \leqslant 0 \tag{2.5}$$

Theorem 2.1 applies to hypercubic $\boldsymbol{\varphi}^4$ models. The physical meaning of (2.3)–(2.5) is that the 1-components of the spins are positively correlated among themselves, but negatively correlated with the 2-components. One special case of (2.5) is the inequality $u_4^{1122} \leqslant 0$, a zeroth-order skeleton inequality for the mixed u_4 function.

A complementary result to BM inequalities is the following proposition.

Proposition 2.2. (Reverse Bricmont–Monroe inequalities.) Make the same assumptions as in the previous theorem except that now $a_{rp} \leqslant 0$ if both r and p are different from zero, and the ferromagnetic interaction can be more general, namely

$$Z=\int \exp\left(\sum_{K,K'} J_{KK'} \varphi_1^K \varphi_2^{K'}\right) \prod_{x \in L} d\nu_x(\varphi_{1x}, \varphi_{2x})$$

where $J_{KK'} \geqslant 0$ for all multi-indices K, K'. Then for any multi-indices A and B

$$\text{(rBM-I)} \qquad \langle \varphi_1^A; \varphi_1^B \rangle \geqslant 0 \tag{2.6}$$

$$\text{(rBM-II)} \qquad \langle \varphi_2^A; \varphi_2^B \rangle \geqslant 0 \tag{2.7}$$

$$\text{(rBM-III)} \qquad \langle \varphi_1^A; \varphi_2^B \rangle \geqslant 0 \tag{2.8}$$

Proof. To see this, we consider the terms $a_{rp} \varphi_1^{2r} \varphi_2^{2p}$ for r, $p \neq 0$ as the ferromagnetic part of a Hamiltonian of two one-component fields. Then we apply Griffiths' second inequality for one-component systems. ∎

Remarks. 1. It is worth noting that the "r" in reversed or (as will be seen in the next section) the "rot" in rotated refers more to the type of measure than to the inequality.

2. Proposition 2.2 applies to hypercubic φ^4 models with $R \leqslant 0$.

2.2. (ρ, θ)-Type Inequalities

Let $\mathscr{F}$ be the set of multinomials in $\{\cos(m_1 \theta_1 + \cdots + m_l \theta_l);\ m_i \in Z\}$ with nonnegative coefficients. Let $\mathscr{G}$ be the set of multinomials in

$$\left\{ \prod_{j=1}^{l} h_j(r_j) \colon h_j(r) \geqslant 0 \text{ nondecreasing on } [0, \infty) \text{ and } O(e^{br^2}) \text{ for some } b > 0 \right\}$$

with nonnegative coefficients. Let $\mathscr{Q}$ be the family of functions on $(\mathbf{R}^2)^{|L|}$ which (in polar coordinates) are multinomial of functions from $\mathscr{F}$ and $\mathscr{G}$ with nonnegative coefficients.

Example. In polar coordinates

$$\begin{aligned} -H &= -\lambda[\varphi_1^4 + \varphi_2^4 + R\varphi_1^2 \varphi_2^2] - \frac{a_1}{2} \varphi_1^2 - \frac{a_2}{2} \varphi_2^2 - b\varphi_1 \varphi_2 \\ &= -\left[\frac{\lambda}{4}\left(3 + \frac{R}{2}\right) \rho^4 + \frac{1}{4}(a_1 + a_2)\rho^2\right] - \frac{\lambda}{4}\left(1 - \frac{R}{2}\right) \rho^4 \cos 4\theta \\ &\quad - \frac{1}{4} \rho^2 (a_1 - a_2) \cos 2\theta - 2b \sin 2\theta \end{aligned} \tag{2.9}$$

Since the first term of (2.9) is isotropic, it can be considered as part of the single spin measure. The remainder of the Hamiltonian is in $-\mathscr{Q}$ if $R \geqslant 2$, $a_1 \leqslant a_2$, and $b = 0$.

Theorem 2.3. (Dunlop–Newman inequalities[3]). Suppose $\{\boldsymbol{\varphi}_x : x \in \mathbf{L}\}$ are random two-dimensional vectors whose joint probability is given by

$$Z^{-1} \exp\left(\sum_{x,y} J^1_{xy} \varphi_{1x}\varphi_{1y} + J^2_{xy} \varphi_{2x}\varphi_{2y} + \sum_x \mathbf{h}_x \cdot \boldsymbol{\varphi}_x\right) \prod_x dv_x(\boldsymbol{\varphi}_x) \quad (2.10)$$

where the *a priori* measure is totally even such that

$$\int \exp(b\,|\boldsymbol{\varphi}|^2)\, dv_x(\boldsymbol{\varphi}) < \infty \qquad \forall b, x$$

if $\mathbf{h}_x = (h_x, 0)$ with $h_x \geqslant 0$, $|J^2_{xy}| \leqslant J^1_{xy}$ for $x \neq y$, and $J^2_{xx} \leqslant J^1_{xx}$ for all x, y. Then for any F, $G \in \mathcal{Q}$,

$$\langle F(\boldsymbol{\varphi}_1, ..., \boldsymbol{\varphi}_l)\rangle \geqslant 0 \quad (2.11)$$

$$\langle F(\boldsymbol{\varphi}_1, ..., \boldsymbol{\varphi}_l);\, G(\boldsymbol{\varphi}_1, ..., \boldsymbol{\varphi}_l)\rangle \geqslant 0 \quad (2.12)$$

Proof. (Sketch) The proof essentially consists in showing that the terms in the exponential on (2.10) belong to the class $\mathcal{Q}$ and applying Proposition 3 of Ginibre.[8] ▮

Remark. A characterization of the class of measures for which BM inequalities hold was developed in an unpublished paper by Ellis and Newman.[9] This work is very closely related to that given by Ellis and Newman in ref. 10 and Ellis *et al.* in ref. 11.

3. ROTATED INEQUALITIES

Here we discuss the process of rotating correlation inequalities for two-component hypercubic systems. Let us rotate the variables $\{\varphi_1, \varphi_2\}$ by 45°. That is, let us define

$$\varphi'_1 = (\varphi_1 + \varphi_2)/\sqrt{2}; \qquad \varphi'_2 = (\varphi_1 - \varphi_2)/\sqrt{2}$$

Then

$$\begin{aligned} H &\equiv \lambda[\varphi_1^4 + \varphi_2^4 + R\varphi_1^2\varphi_2^2] + \frac{a_1}{2}\varphi_1^2 + \frac{a_2}{2}\varphi_2^2 + b\varphi_1\varphi_2 \\ &= \frac{\lambda}{2}\left(1 + \frac{R}{2}\right)[\varphi_1'^4 + \varphi_2'^4] + \lambda\left(3 - \frac{R}{2}\right)\varphi_1'^2\varphi_2'^2 + \frac{a_1 + a_2}{4}(\varphi_1'^2 + \varphi_2'^2) \\ &\quad + \frac{(a_1 - a_2)}{2}\varphi'_1\varphi'_2 + \frac{b}{2}(\varphi_1'^2 - \varphi_2'^2) \end{aligned} \quad (3.1)$$

$$\equiv \lambda'[\varphi_1'^4 + \varphi_2'^4 + R'\varphi_1'^2\varphi_2'^2] + \frac{a'_1}{2}\varphi_1'^2 + \frac{a'_2}{2}\varphi_2'^2 + b'\varphi'_1\varphi'_2 \quad (3.2)$$

where

$$\lambda' = \frac{\lambda}{2}\left(1 + \frac{R}{2}\right) \tag{3.3}$$

$$R' = \frac{6 - R}{1 + R/2} \tag{3.4}$$

$$a_1' = \frac{a_1 + a_2}{2} + b \tag{3.5}$$

$$a_2' = \frac{a_1 + a_2}{2} - b \tag{3.6}$$

$$b' = \frac{a_1 - a_2}{2} \tag{3.7}$$

Remarks. 1. Notice that $a_1 \leqslant a_2$ iff $b' \leqslant 0$, and $b = 0$ iff $a_1' = a_2'$.

2. Notice that (3.4) can equivalently be written as

$$\frac{1}{R+6} + \frac{1}{R'+6} = \frac{1}{4} \tag{3.8}$$

which makes clear the duality between R and R'. Some important special cases are

$$R = -2 \to R' = \infty$$
$$R = 0 \to R' = 6$$
$$R = 2 \to R' = 2$$
$$R = 6 \to R' = 0$$
$$R = \infty \to R' = -2$$

which have the following interpretation for $b = 0$:

$R = -2$ (pure $\varphi_1^2 \varphi_2^2$ model, 45° rotated)

$R = 0$ (decoupled one-component φ^4 models)

$R = 2$ and $a_1 = a_2$ (isotropic $|\boldsymbol{\varphi}|^4$ model)

$R = 6$ (decoupled one-component φ^4 models, 45° rotated)

$R = \infty$ (pure $\varphi_1^2 \varphi_2^2$ model)

Let us consider a very special case of Dunlop–Newman inequalities (Theorem 2.3):

$$\text{(vsD-N)} \qquad \langle \varphi_{1x}^2; \varphi_{1y}^2 + \varphi_{2y}^2 \rangle \geqslant 0 \tag{3.9}$$

which is valid for $R \geqslant 2$, $a_1 \leqslant a_2$, and $b = 0$. These conditions can be easily seen by expressing the Hamiltonian H in polar coordinates and checking the conditions on the coefficients in the Hamiltonian in order that its non-isotropic part belongs to the class $-\mathscr{Q}$. We have already done this in the example of Section 2.

Proposition 3.1. (Rotated very special Dunlop–Newman inequalities.) Consider a system with joint probability measure given by (1.2), where $R \in [-2, 2]$, $a_1 = a_2$, and $b \leqslant 0$. Then we have

$$\langle \varphi_{1x}^2 + \varphi_{2x}^2 + 2\varphi_{1x}\varphi_{2x}; \varphi_{1y}^2 + \varphi_{2y}^2 \rangle \geqslant 0 \tag{3.10}$$

In particular, if we further restrict to $b = 0$, we have

$$\text{(rotvsD-N)} \qquad \langle \varphi_{1x}^2; \varphi_{1y}^2 + \varphi_{2y}^2 \rangle \geqslant 0 \tag{3.11}$$

Proof. Let us translate (vsD-N) to the primed variables:

$$\left\langle \left(\frac{\varphi_1' + \varphi_2'}{\sqrt{2}} \right)_x^2; \left(\frac{\varphi_1' + \varphi_2'}{\sqrt{2}} \right)_y^2 + \left(\frac{\varphi_1' - \varphi_2'}{\sqrt{2}} \right)_y^2 \right\rangle \geqslant 0$$

that is,

$$\langle \varphi_{1x}'^2 + \varphi_{2x}'^2 + 2\varphi_{1x}'\varphi_{2x}'; \varphi_{1y}'^2 + \varphi_{2y}'^2) \geqslant 0 \tag{3.12}$$

Translating the conditions $R \geqslant 2$, $a_1 \leqslant a_2$, and $b = 0$ to the primes, we get $R' \leqslant 2$, $b' \leqslant 0$, and $a_1' = a_2'$, respectively. Dropping primes, we get (3.10). In the case $b = 0$ we have the symmetries $\varphi_1' \leftrightarrow \varphi_2'$, $\varphi_1' \leftrightarrow -\varphi_1'$, and $\varphi_2' \leftrightarrow -\varphi_2'$. For this particular case (3.12) becomes

$$\langle \varphi_{1x}'^2; \varphi_{1y}'^2 + \varphi_{2y}'^2 \rangle \geqslant 0$$

Finally dropping primes, we get (3.11). ∎

Remark. It is amusing to note that in the symmetric case $a_1 = a_2$ and $b = 0$, the rotated (vsD-N) inequality (3.11) is identical to the original (vsD-N) inequality (3.9); what this proof shows is that it is valid for R, not just $R \geqslant 2$. Another way to prove vsD-N for $R \in [-2, 0]$ and $b \leqslant 0$ for models with Hamiltonian of type (3.1) is using reverse BM inequalities.

Let us consider $\langle \varphi_1^4; \varphi_{1z}^2 - \varphi_{2z}^2 \rangle \geqslant 0$ and $\langle \varphi_2^4; \varphi_{2z}^2 - \varphi_{1z}^2 \rangle \geqslant 0$. Notice that these inequalities are straightforward consequences of BM inequalities for $R \in [0, \infty]$. The interesting fact is that they can be obtained and extended for all R in the case $A = \{x, y\}$, $a_1 = a_2$, and $b = 0$ from the rotated Griffiths' first inequality.

Proposition 3.2. Consider a system with joint probability measure given by (1.2), where $a_1 \leqslant a_2$, $b = 0$, and $R \in [-2, \infty]$. Then for all x, y, and z in $\mathbf{L}$

$$\langle \varphi_{1x}\varphi_{1y} - \varphi_{2x}\varphi_{2y};\, \varphi_{1z}^2 - \varphi_{2z}^2 \rangle \geqslant 0 \tag{3.13}$$

In particular, if we further restrict to $a_1 = a_2$, we have

$$\text{(rotG-I)} \qquad \langle \varphi_{1x}\varphi_{1y};\, \varphi_{1z}^2 - \varphi_{2z}^2 \rangle \geqslant 0 \tag{3.14}$$

Proof. From Griffiths' first inequality, which holds for a system described by (1.2) with $b \leqslant 0$ and all a_1, a_2, and R, we have

$$\langle \varphi_{1x}\varphi_{2y}\varphi_{1z}\varphi_{2z} \rangle \geqslant 0$$

Now we translate to the primed variables and group the variables in the following way:

$$\begin{aligned} &\langle \varphi'_{1x}\varphi'_{1y}(\varphi'^2_{1z} - \varphi'^2_{2z}) \rangle + \langle \varphi'_{2x}\varphi'_{2y}(\varphi'^2_{2z} - \varphi'^2_{1z}) \rangle \\ &\quad - \langle \varphi'_{1x}\varphi'_{2y}(\varphi'^2_{1z} - \varphi'^2_{2z}) \rangle - \langle \varphi'_{2x}\varphi'_{1y}(\varphi'^2_{2z} - \varphi'^2_{1z}) \rangle \geqslant 0 \end{aligned} \tag{3.15}$$

Translating the condition $b \leqslant 0$ to the primes, we get $a'_1 \leqslant a'_2$. If we impose the additional condition $b' = 0$, then the symmetries $\varphi'_1 \to -\varphi'_1$, $\varphi'_2 \to -\varphi'_2$ imply that the last two terms of (3.15) vanish. Dropping primes, we get (3.13). If, finally, we further restrict to $a'_1 = a'_2$, the symmetry $\varphi'_1 \leftrightarrow \varphi'_2$ implies that the first two terms in (3.15) are equal. Dropping primes, we get (3.14). ∎

One interesting result concerning rotated correlation inequalities is the Gaussian inequality for multicomponent rotators proved by Bricmont.[12]

One straightforward conclusion from Bricmont's derivation for u_4 functions is

$$u_4^{1111} \leqslant 0 \qquad \text{whenever} \quad 0 \leqslant R \leqslant 6, \quad b = 0, \quad a_1 = a_2 \tag{3.16}$$

For this special case is easy to obtain (3.16) from the process of rotation. That is, let us rotate $u_4^{1122} \leqslant 0$ for $R \geqslant 0$ and their respective permutations by 45°. Then we get the rotated BM-III inequality

$$u_4^{1111} - \tfrac{1}{3}(u_4^{1122} + u_4^{1212} + u_4^{1221}) \leqslant 0 \qquad \text{whenever} \quad R \leqslant 6, \quad b = 0, \quad a_1 = a_2 \tag{3.17}$$

Combining with $u_4^{1122} \leqslant 0$, $R \geqslant 0$, and their permutations, we get

$$u_4^{1111} \leqslant 0 \qquad \text{whenever} \quad b=0, \quad a_1 = a_2, \quad 0 \leqslant R \leqslant 6 \tag{3.18}$$

which is precisely (3.16).

4. THE RANDOM WALK REPRESENTATION

The random walk representation for two-component systems (with $b=0$) can be deduced following a similar analysis to the one-component case discussed.[13,14] There are minor differences; for example, now we have two local times (s and t) instead of one (t). We present here the conclusions of the random walk representation for two-component systems.

The two-point function is given by

$$\langle \varphi_{1x} \varphi_{1y} \rangle = \sum_{\omega:\, x \to y} J^{\omega} \int dv_{\omega}(t)\, \mathscr{Z}(t, 0) \tag{4.1}$$

where

$$\mathscr{Z}(t, s) \equiv Z^{-1} \int \exp[\tfrac{1}{2}(\boldsymbol{\varphi}, J\boldsymbol{\varphi})] \prod_x g_x(\varphi_{1x}^2 + 2t_x, \varphi_{2x}^2 + 2s_x)\, d\varphi_{1x}\, d\varphi_{2x} \tag{4.2}$$

Z is the usual partition function and g_x is defined by $\mathscr{Z}(0, 0) = 1$. The four-point functions are given by:

(i) Nonmixed fields

$$\langle \varphi_{1x_1} \varphi_{1x_2} \varphi_{1x_3} \varphi_{1x_4} \rangle$$
$$= \sum_{\omega:\, x_1 \to x_2} J^{\omega} \int dv_{\omega}(t)\, \mathscr{Z}(t, 0) \langle \varphi_{1x_3} \varphi_{1x_4} \rangle_{t,0} + \text{two permutations} \tag{4.3}$$

and similarly with the one-component case $\langle \cdot \rangle_{t,s}$ means normalized expectation with respect to the measure $Z(t, s)$ given by the integral in (4.2).

(ii) Mixed fields

$$\langle \varphi_{1x_1} \varphi_{1x_2} \varphi_{2x_3} \varphi_{2x_4} \rangle = \sum_{\omega:\, x_1 \to x_2} J^{\omega} \int dv_{\omega}(t)\, \mathscr{Z}(t, 0) \langle \varphi_{2x_3} \varphi_{2x_4} \rangle_{t,0} \tag{4.4a}$$

$$= \sum_{\omega:\, x_3 \to x_4} J^{\omega} \int dv_{\omega}(s)\, \mathscr{Z}(0, s) \langle \varphi_{1x_1} \varphi_{1x_2} \rangle_{0,s} \tag{4.4b}$$

The splitting-of-paths lemma is completely similar to the one-component systems.[15]

5. MONOTONICITY OF $\langle\varphi_1^A\rangle_{t,s}$ AND $\langle\varphi_2^A\rangle_{t,s}$ IN t, s

We will see shortly (Section 6) that the analysis of the skeleton inequalities for u_4 functions requires the knowledge of the monotonicity of $\langle\varphi_1^A\rangle_{t,s}$ and $\langle\varphi_2^A\rangle_{t,s}$ in t, s variables. In ref. 1, we develop an exhaustive study of the monotonicity for the two-component hypercubic φ^4 models with $b=0$. There we divided our analysis into two cases: (1) $R\in[-2,\infty)$, and (2) $R=\infty$ (i.e., $\lambda_1=0$).

5.1. $R\in[-2,\infty)$

For this case

$$g_x(\varphi_1^2,\varphi_2^2)=\exp\left\{-\lambda[\varphi_1^4+\varphi_2^4+R\varphi_1^2\varphi_2^2]-\frac{a_1}{2}\varphi_1^2-\frac{a_2}{2}\varphi_2^2\right\} \tag{5.1}$$

where λ here denotes λ_1. Then

$$g_x(\varphi_1^2+2t,\varphi_2^2+2s)=\exp\left\{-\lambda[\varphi_1^4+\varphi_2^4+R\varphi_1^2\varphi_2^2]-\frac{\alpha_1}{2}\varphi_1^2-\frac{\alpha_2}{2}\varphi_2^2 - (4\lambda t^2+4\lambda s^2+4tsR+a_1t+a_2s)\right\} \tag{5.2}$$

where $\alpha_1/2\equiv a_1/2+4\lambda t+2\lambda sR$ and $\alpha_2/2\equiv a_2/2+4\lambda s+2\lambda tR$. In a similar way as in the one-component case, we notice that the effect of the t and s variables is to add space-dependent mass terms $(4\lambda t+2\lambda sR)\varphi_1^2$ and $(4\lambda s+2\lambda tR)\varphi_2^2$ to the Hamiltonian. Now in order to find the monotonicity of the moments in t, s variables we study the sign of the derivatives of the moments with respect to t and s. That is,

$$\frac{\partial}{\partial t_z}\langle\varphi_1^A\rangle_{t,s}=-4\lambda\left\langle\varphi_1^A;\varphi_{1z}^2+\frac{R}{2}\varphi_{2z}^2\right\rangle_{t,s} \tag{5.3}$$

$$\frac{\partial}{\partial t_z}\langle\varphi_2^A\rangle_{t,s}=-4\lambda\left\langle\varphi_2^A;\varphi_{1z}^2+\frac{R}{2}\varphi_{2z}^2\right\rangle_{t,s} \tag{5.4}$$

$$\frac{\partial}{\partial s_z}\langle\varphi_1^A\rangle_{t,s}=-4\lambda\left\langle\varphi_1^A;\frac{R}{2}\varphi_{1z}^2+\varphi_{2z}^2\right\rangle_{t,s} \tag{5.5}$$

$$\frac{\partial}{\partial s_z}\langle\varphi_2^A\rangle_{t,s}=-4\lambda\left\langle\varphi_2^A;\frac{R}{2}\varphi_{1z}^2+\varphi_{2z}^2\right\rangle_{t,s} \tag{5.6}$$

Remark. Notice that it is enough to study two cases [i.e., (5.3), (5.4)] and by analogy one determines the other two.

Thus we concentrate our attention on finding the sign of

$$\langle \varphi_1^A; \varphi_{1z}^2 + \kappa \varphi_{2z}^2 \rangle_{t,s} \tag{5.7}$$

and

$$\langle \varphi_2^A; \varphi_{1z}^2 + \kappa \varphi_{2z}^2 \rangle_{t,s} \tag{5.8}$$

where $\kappa \in \mathbf{R}$ and the expectation $\langle \cdot \rangle_{t,s}$ is R dependent. To do this, we studied the signs of (5.7), (5.8) in the plane (R, κ). Here we concentrate on those regions of the plane that intersect the case $\kappa = R/2$, which is our main interest (the complete study can be found in ref. 1).

Study of (5.7) for $A = \{x, y\}$

I. $R \leqslant 0$, $\kappa \geqslant 0$. From the decomposition

$$\langle \varphi_{1x}\varphi_{1y}; \varphi_{1z}^2 + \kappa \varphi_{2z}^2 \rangle_{t,s} = \langle \varphi_{1x}\varphi_{1y}; \varphi_{1z}^2 \rangle_{t,s} + \kappa \langle \varphi_{1x}\varphi_{1y}; \varphi_{2z}^2 \rangle_{t,s} \tag{5.9}$$

We conclude that $(5.7) \geqslant 0$ from rBM inequalities.

II. $R \geqslant 0$, $\kappa \leqslant 0$. Using the same decomposition as in region I, we conclude that $(5.7) \geqslant 0$ from BM inequalities.

III. $R \geqslant 2$, $\kappa \leqslant 1$. From the decomposition

$$\begin{aligned} \langle \varphi_{1x}\varphi_{1y}; \varphi_{1z}^2 + \kappa \varphi_{2z}^2 \rangle_{t,s} &= \langle \varphi_{1x}\varphi_{1y}; \varphi_{1z}^2 + \varphi_{2z}^2 \rangle_{t,s} \\ &\quad + (\kappa - 1)\langle \varphi_{1x}\varphi_{1y}; \varphi_{2z}^2 \rangle_{t,s} \end{aligned} \tag{5.10}$$

We conclude that $(5.7) \geqslant 0$ from vsD-N and BM inequalities. Notice that in this case we have the restriction $\alpha_1 \leqslant \alpha_2$. At the moment we do not know whether $(5.7) \geqslant 0$ holds in general in region III, namely for $\alpha_1 > \alpha_2$.

Remark. The condition $\alpha_1 \leqslant \alpha_2$ is equivalent to $(a_1 - a_2) + 4\lambda(R-2)(s_i - t_i) \leqslant 0$ for all i. Sufficient conditions for this to hold are:

(a) $R = 2$, $a_1 \leqslant a_2$.

(b) $R \leqslant 2$, $a_1 \leqslant a_2$, $s_i \geqslant t_i$ for all i (e.g., $t_i = 0$).

(c) $R \geqslant 2$, $a_1 \leqslant a_2$, $s_i \leqslant t_i$ for all i (e.g., $s_i = 0$).

Conclusions Regarding (5.7) for Our Case of Main Interest $\kappa = R/2$:

(a) $R = 0$, $(5.7) \geqslant 0$ from regions I, II.

(b) $R = 2$, $(5.7) \geqslant 0$ from region III if $\alpha_1 \leqslant \alpha_2$ (i.e., $s_i \leqslant t_i$ for all sites i).

(c) $R > 2$, a general proof that $(5.7) \geqslant 0$ or $(5.7) \leqslant 0$ is hopeless from counterexample (the one-site model with probability measure given by (1.2); computations can be found in Appendix A of ref. 1).

Study of (5.8) for $A = \{x, y\}$

I. $R \leqslant 0$, $\kappa \geqslant 0$. From the decomposition

$$\langle \varphi_{2x}\varphi_{2y};\, \varphi_{1z}^2 + \kappa\varphi_{2z}^2 \rangle_{t,s} = \langle \varphi_{2x}\varphi_{2y};\, \varphi_{1z}^2 \rangle_{t,s} + \kappa \langle \varphi_{2x}\varphi_{2y};\, \varphi_{2z}^2 \rangle_{t,s}$$

we conclude that $(5.8) \geqslant 0$ from rBM inequalities.

II. $R \geqslant 2$, $\kappa \geqslant 1$. From the decomposition

$$\langle \varphi_{2x}\varphi_{2y};\, \varphi_{1z}^2 + \kappa\varphi_{2z}^2 \rangle_{t,s} = \langle \varphi_{2x}\varphi_{2y};\, \varphi_{1z}^2 + \varphi_{2z}^2 \rangle_{t,s} + (\kappa - 1) \langle \varphi_{2x}\varphi_{2y};\, \varphi_{2z}^2 \rangle_{t,s}$$

we conclude that $(5.8) \geqslant 0$ from vsD-N and BM inequalities. We have the restriction $\alpha_1 \leqslant \alpha_2$. Three sufficient conditions for $\alpha_1 \leqslant \alpha_2$ to hold were given above [region III of (5.7)].

Conclusions Regarding (5.8) for Our Case of Main Interest $\kappa = R/2$:

(a) $R = 0$, $(5.8) \geqslant 0$ from region I.

(b) $R \geqslant 2$, $(5.8) \geqslant 0$ from region II.

5.2. $R = \infty\, (\lambda_1 = 0)$

For this case

$$g_x(\varphi_1^2, \varphi_2^2) = \exp\left(-\lambda_1^2 \varphi_2^2 - \frac{a_1}{2}\varphi_1^2 - \frac{a_2}{2}\varphi_2^2 \right) \tag{5.11}$$

where here λ denotes λ_2. Then

$$g_x(\varphi_1^2 + 2t, \varphi_2^2 + 2s) = \exp\left[-\lambda\varphi_1^2\varphi_2^2 - \frac{\alpha_1}{2}\varphi_1^2 - \frac{\alpha_2}{2}\varphi_2^2 - (4\lambda ts + a_1 t + a_2 s) \right] \tag{5.12}$$

where $\alpha_1/2 = a_1/2 + 2s\lambda$ and $\alpha_2 = a_2/2 + 2t\lambda$. In this case as in the one-component case the effect of the t and s variables is to add space-dependent mass terms $(2s\lambda)\,\varphi_1^2$ and $(2t\lambda)\,\varphi_2^2$ to the Hamiltonian. In order to find the monotonicity of the moments in the t, s variables, we study the sign of the derivatives of the moments with respect to t (since it is enough to study just these cases and by analogy determine the s cases). That is,

$$\frac{\partial}{\partial t_z}\langle\varphi_1^A\rangle_{t,s} = -2\lambda\langle\varphi_1^A;\varphi_{2z}^2\rangle_{t,s} \tag{5.13}$$

$$\frac{\partial}{\partial t_z}\langle\varphi_2^A\rangle_{t,s} = -2\lambda\langle\varphi_2^A;\varphi_{2z}^2\rangle_{t,s} \tag{5.14}$$

From BM inequalities we know that

$$\langle\varphi_1^A;\varphi_{2z}^2\rangle_{t,s}\leqslant 0$$

$$\langle\varphi_2^A;\varphi_{2z}^2\rangle_{t,s}\geqslant 0$$

6. SKELETON INEQUALITIES FOR u_4 FUNCTIONS

Skeleton inequalities for two-component systems present interesting differences (as we will see shortly) compared with the one-component systems.

By (4.1) and (4.3) and simplifying the subscripts and superscripts we can write

$$u_4^{1111}(1,2,3,4)=F(1,2\,|\,3,4)+F(1,3\,|\,2,4)+F(1,4\,|\,2,3) \tag{6.1}$$

where

$$F(1,2\,|\,3,4)\equiv\sum_{\omega:\,x_1\to x_2} J^\omega\int dv_\omega(t)\,\mathscr{Z}(t,0)[\langle\varphi_{1x_3}\varphi_{1x_4}\rangle_{t,0}-\langle\varphi_{1x_3}\varphi_{1x_4}\rangle_{0,0}] \tag{6.2}$$

Since J^ω, dv_ω, and $\mathscr{Z}(t,0)$ are all nonnegative, the sign of u_4^{1111} will depend on the sign of the brackets, that is, on the monotonicity of the expectation $\langle\varphi_1^A\rangle_{t,0}$ with respect to t.

In a similar way for the mixed case, we have

$$u_4^{1122}(1,2,3,4)=\sum_{\omega:\,x_1\to x_2} J^\omega\int dv_\omega(t)\,\mathscr{Z}(t,0)[\langle\varphi_{2x_3}\varphi_{2x_4}\rangle_{t,0}-\langle\varphi_{2x_3}\varphi_{2x_4}\rangle_{0,0}] \tag{6.3}$$

The sign of u_4^{1122} will depend on the sign of the brackets, that is, on the monotonicity of the expectation $\langle\varphi_2^A\rangle_{t,0}$ with respect to t.

6.1. For $A=\{x,y\}$ and Specializing to $\kappa=R/2$

1. $\langle\varphi_1^A\rangle_{t,0}$ is monotonic decreasing in t for $R=0$ with no restrictions on α_1, α_2 (hence with no restrictions on a_1, a_2). Therefore from (6.2) we have

$$u_4^{1111}\leqslant 0 \qquad \text{if} \quad R=0 \ \ (a_1, a_2 \text{ arbitrary}) \tag{6.4}$$

2. $\langle \varphi_1^A \rangle_{t,0}$ is monotonic decreasing in t for $R=2$ with the restriction $\alpha_1 \leqslant \alpha_2$ (hence with the restriction $a_1 \leqslant a_2$). Therefore we have

$$u_4^{1111} \leqslant 0 \qquad \text{if} \quad R=2, \quad a_1 \leqslant a_2 \tag{6.5}$$

3. $\langle \varphi_2^A \rangle_{t,0}$ is monotonic decreasing in t for $R \geqslant 2$ with the restriction $\alpha_1 \leqslant \alpha_2$ (hence with the restriction $a_1 \leqslant a_2$). Therefore we have

$$u_4^{1122} \leqslant 0 \qquad \text{if} \quad R \geqslant 2, \quad a_1 \leqslant a_2 \tag{6.6}$$

6.2. For any Multi-Index A and Specializing to $\kappa = R/2$

4. $\langle \varphi_1^A \rangle_{t,s}$ is monotonic increasing in t for $R=\infty$ without restrictions on α_1, α_2 (hence without restriction on a_1, a_2). Therefore we have

$$u_4^{1111} \geqslant 0 \qquad \text{if} \quad R=\infty \;\; (a_1, a_2 \text{ arbitrary}) \tag{6.7}$$

5. $\langle \varphi_2^A \rangle_{t,s}$ is monotonic decreasing in t for $R=\infty$ without restriction in α_1, α_2 (hence without restriction on a_1, a_2). Therefore we have

$$u_4^{1122} \leqslant 0 \qquad \text{if} \quad R=\infty \;\; (a_1, a_2 \text{ arbitrary}) \tag{6.8}$$

We summarize the above result in the following proposition:

Proposition 6.1. Consider a system with joint probability measure given by (1.2), where $b=0$. Then for all x_1, x_2, x_3, and x_4 in **L** we have:

(i) $u_4^{1111}(x_1, x_2, x_3, x_4) \leqslant 0$ if $R=0$ and a_1, a_2 arbitrary
(ii) $u_4^{1111}(x_1, x_2, x_3, x_4) \leqslant 0$ if $R=2$ and $a_1 \leqslant a_2$
(iii) $u_4^{1122}(x_1, x_2, x_3, x_4) \leqslant 0$ if $R \geqslant 2$ and $a_1 \leqslant a_2$
(iv) $u_4^{1111}(x_1, x_2, x_3, x_4) \geqslant 0$ if $R=\infty$ and a_1, a_2 arbitrary
(v) $u_4^{1122}(x_1, x_2, x_3, x_4) \leqslant 0$ if $R=\infty$ and a_1, a_2 arbitrary

Remarks. 1. (6.4) is just the Lebowitz inequality for one-component models. Moreover, (6.4)–(6.5) is a special case of the Gaussian inequality for two-component rotator proved in ref. 12 for $a_1 = a_2$ and $0 \leqslant R \leqslant 6$ by different techniques. Note, however, that here (6.5) is proven whenever $a_1 \leqslant a_2$.

2. (6.6) and (6.8) are results already known from BM inequalities (actually BM and rBM inequalities prove $u_4^{1122} \leqslant 0$ for all $R \geqslant 0$ and $u_4^{1122} \geqslant 0$ for $R \leqslant 0$, respectively).

3. The regions where we know the monotonicity of $\langle \varphi_1^A \rangle_{t,0}$ and $\langle \varphi_2^A \rangle_{t,0}$ intercept only in $R=2$ and $R=0$.

7. CONCLUSIONS

The proof of the monotonicity of the expectation of the moments of the fields in the variable t was a problem that we were unable to solve completely. In this matter we reached the following conclusions related to the search of first- and second-order skeleton inequalities in ref. 1:

1. The impossibility of knowing the monotonicity of $\langle \varphi_1^4 \rangle_{t,0}$ in t for $2 < R < \infty$ will be translated into the impossibility of having first- and second-order skeleton inequalities for $R \in (2, \infty)$.

2. Our inability to prove the monotonicity of $\langle \varphi_1^4 \rangle_{t,0}$ and $\langle \varphi_2^4 \rangle_{t,0}$ for $R < 2$, $R \neq 0$ is translated as the inability to prove second-order skeleton inequalities. However, we emphasize that we have no counterexample to the monotonicity of $\langle \varphi_1^4 \rangle_{t,0}$ and $\langle \varphi_2^4 \rangle_{t,0}$ in this region. Thus, if future work should succeed in proving this monotonicity, first- and second-order skeleton inequalities for u_4 functions would immediately follow.

3. The impossibility of proving skeleton inequalities of first and second order for the case $R = \infty$ was mainly due to the fact that the random walk representation method used in ref. 15 did not work for this case.

ACKNOWLEDGMENTS

I thank Alan D. Sokal for his support and many useful conversations during the realization of this work. I also thank Profs. James Glimm and Monroe Donsker for their encouragement and stimulating lectures during my time at the Courant Institute.

REFERENCES

1. J. L. Soria, Ph.D. thesis, Courant Institute, NYU, unpublished (December 1987).
2. J. Bricmont, *Ann. Sci. Bruxelles* **90**:245–252 (1976).
3. F. Dunlop and C. Newman, *Commun. Math. Phys.* **44**:223–235 (1975).
4. J. L. Monroe, *J. Math. Phys.* **16**:1809–1812 (1975).
5. F. Dunlop, *Commun. Math. Phys.* **49**:247–256 (1976).
6. H. Kunz, C. E. Pfister, and P. Vuillermont, *Phys. Lett.* **54**:428–430 (1976).
7. J. Bricmont, Ph.D. thesis, Université Catholique de Louvain, Belgium, unpublished (1975).
8. J. Ginibre, *Commun. Math. Phys.* **16**:310–328 (1970).
9. R. S. Ellis and C. M. Newman, Necessary and sufficient conditions for plane rotators systems, unpublished (1976).
10. R. S. Ellis and C. M. Newman, *Trans. Math. Soc.* **237**:83–99 (1978).
11. R. S. Ellis, J. L. Monroe, and C. Newman, *Commun. Math. Phys.* **46**:167–182 (1976).
12. J. Bricmont, *J. Stat. Phys.* **17**:289–300 (1977).
13. D. C. Brydges, J. Fröhlich, and T. Spencer, *Commun. Math. Phys.* **83**:123–150 (1982).
14. D. Brydges, in *Gauge Theories: Fundamentals Interactions and Rigorous Results*, P. Dita, V. Georgescu, and R. Purice, eds. (Birkhäuser, Boston, 1982).
15. D. Brydges, J. Fröhlich, and A. Sokal, *Commun. Math. Phys.* **91**:117–139 (1983).